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Black hole entropy

P Mitra*

Saha Institute of Nuclear Physics, Block AF, Bidhannagar,
Calcutta-700 064, India

Abstract The idea that a black hole may have an entropy proportional to the horizon area was introduced in the seventies but a microscopic understanding was not available for a long time Some progress has been made in the last couple of years

Keywords Black hole, entropy, horizon area

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1. Introduction

A black hole is classically thought of as a region of intense gravitational field from which no form of energy – not even light – even leak out. The best known example is the Schwarzschild black hole solution of Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \tag{1}$$

in empty space ($T_{\mu\nu} = 0$). It is described by the metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2}$$

where the parameter M refers to the mass of a point source of gravitation located at the centre of coordinates. The spacetime has a *horizon* at $r = 2M$, which is a singularity of this coordinate system, but the curvature is not singular there, and regular coordinates (discovered by *Kruskal*) may be chosen. There is, however, a curvature singularity at $r = 0$, the location of the point source.

Another example is the Reissner – Nordström solution of the Einstein – Maxwell equations. The metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3}$$

*e-mail mitra@tnp.saha.ernet.in

e^a being a *triad*, ω^a the spin connection and l being related to the cosmological constant Λ by

$$\Lambda = \frac{1}{l^2}. \tag{14}$$

The SO (2, 1) gauge theory has a level

$$k = \frac{l\sqrt{2}}{8G}. \tag{15}$$

The diffeomorphism invariance of 2 + 1 gravity corresponds to the gauge invariance of the Chern-Simons theory. Such a theory is topological and when the gauge degrees of freedom are removed, no physical degrees of freedom are left unless the manifold on which it is defined has a boundary, and it is on the boundary that the degrees of freedom are situated. They correspond to deformations of the boundary. With the identification of the horizon in a black hole spacetime as such a boundary, the degrees of freedom correspond to deformations of the horizon. For large k, the boundary degrees of freedom in the Chern-Simons theory, which can be described by a Wess-Zumino-Witten action on the boundary, can be seen to be governed by a 6 dimensional bosonic string theory. However, instead of the usual Virasoro conditions $L_n = 0$, only the condition $L_0 = 0$ is to be imposed here because of the boundary, which is invariant under only a restricted class of deformations (rigid rotations). L_0 depends on a number operator N, the spin connection ω and the horizon radius r_+, so that a relation is found between these. ω is supposed to be integrated over, and this has the effect of extremizing the relation with respect to ω. The result is the relation

$$N = \left(\frac{r_+}{4G}\right)^2. \tag{16}$$

In view of this, the number of states is given by the conformal field theory formula

$$\exp\left(\pi\sqrt{6.2}\,N/3\right) = \exp\left(\frac{2\pi r_+}{4G}\right). \tag{17}$$

The corresponding entropy is $\left(\frac{2\pi r_+}{4G}\right)$, which is nothing but the *area* divided by $4G$.

2.1 Extension to 3 + 1 dimensions :

These ideas have to some extent been extended to 3 + 1 dimensions [5]. Although an explicit Chern-Simons-like formulation is lacking, a separation of bulk and boundary degrees of freedom has been made, and further it has been argued that the bulk degrees are irrelevant for the calculation of the entropy of a black hole. The surface degrees are not easy to count. The intersections of the surface with appropriate "spin states" are counted to give a rough estimate and the area law is found to hold upto a proportionality constant which involves an undetermined parameter – the so-called "Immirzi parameter."

3. String theoretic approach

By far the most vocal has been the claim made by string theorists. String theory has for a long time been regarded as a candidate for a reasonable quantum theory of gravity because it contains the Einstein action in a certain sense and is moreover a finite theory, free of the infinities that plague the field theory approach to quantum gravity. Recent progress relates to the study of black holes in string theory.

The fields of heterotic string theory compactified on a 6 dimensional torus are described by an effective action which admits black hole spacetimes as solutions. The extreme ones among these are supposed to correspond to massive states of the string theory. The number of such states corresponding to a single classical spacetime solution may be interpreted as a degeneracy and may be used to calculate the entropy of the (extremal) black hole.

This idea was used [6] for some electrically charged black hole solutions of *zero horizon area*. The degeneracy of the massive string states with the appropriate quantum numbers can be calculated from standard heterotic string theory. This leads to a nonzero entropy, while the area of the horizon is zero. To explain this failure, it was speculated that quantum effects would *stretch the horizon* and give it an area of the right magnitude to agree with the calculated entropy. Shades of *Procrustes* !

Such violent manoeuvres were soon shown to be entirely unnecessary : it was possible [7] to reproduce the expected entropy of a 5 dimensional Reissner Nordström black hole by a string theoretic calculation without any stretching.

Black hole solutions of string theory with nonzero horizon area do not correspond to massive states in the spectrum of elementary excitations – they are nonperturbative solitonic excitations instead ; more generally, they are bound states of elementary solitons and strings. The counting of the solitonic states is difficult in general, but in special cases of BPS saturation, supersymmetry provides assistance. The black hole description applies at strong coupling, but the degeneracy can be calculated at weak coupling because in these cases the number may be taken to be preserved under change of coupling by virtue of supersymmetry. The original calculations have been extended to some more extremal black holes in different dimensions.

Non-extremal black holes have also been considered in this approach [8]. Here there is no supersymmetry to guarantee the preservation of the number of states, but calculations at weak coupling have reproduced the expected entropy at strong coupling for several near-extremal black holes. A simplified version of this approach is described in the next subsection.

It is to be emphasized that all these calculations refer to rather special cases where there is some supersymmetry in the extremal limit, and in the non-extremal extension it is not possible to go far from extremality. The situation in 2 + 1 dimensions was better, because there was no use of super-symmetry and no nearness to extremality was invoked, but of course the Chern-Simons manipulations depend crucially on the 2 + 1 dimensional nature of the system.

3.1 Simplified model for near-extremal black holes :

In this section we seek to demonstrate the use of a one-dimensional gas of massless particles [9] as a simplified version of the string model for the dyonic black holes. The model can work for black holes in any number of dimensions, though it will be used here only for four dimensional black holes. The particles can be either left-moving or right-moving – there is no mixing between the two types. Both bosons and fermions can be present. If the total length of the one-dimensional space is L, the entropy and the energy are given by

$$S = \frac{\pi L}{6\hbar}\left[n_L T_L + n_R T_R\right],$$

$$E = \frac{\pi L}{12\hbar}\left[n_L T_L^2 + n_R T_R^2\right], \tag{18}$$

where $n_L(n_R)$ is the number of left (right)-moving bosons plus half the corresponding number of fermions, because a fermion contributes only a half of the contibution a boson makes to the free energy. In the absence of interactions, the left and right degrees of freedom are independent, and the corresponding temperatures can be different. The effective temperature may be defined by $\left(\frac{\partial S}{\partial E}\right)$, the differentiation being carried out at constant momentum, *i.e.*, constant difference between E_L and E_R. This leads to a temperature

$$T = \frac{2T_L T_R}{T_L + T_R} \tag{19}$$

equal to the harmonic mean of T_l and T_R. If $n_l = n_R = n$, these equations get somewhat simplified and one has

$$E = \frac{\pi nL}{12h}\left(\frac{6hS}{\pi nL}\right)^2 - \frac{6\hbar ST}{\pi nL} \tag{20}$$

To compare these quantities with those for a near-extremal Reissner – Nordström black hole with $r - r \equiv b << Q$, it is necessary to fix two parameters of the gas to be equal to the corresponding parameters of the black hole, and check how the third parameter of the gas compares with that parameter for the black hole. Thus, we could set the temperatures and energies to be equal and compare the entropies. However, as the entropy of the gas enters the energy quadratically, the expression for the entropy in terms of the energy involves a square root. This can be handled, but it is much simpler to fix the temperatures and the entropies and compare the energies. Thus, we put

$$T = T_H(b, Q)$$

$$= \frac{b}{2\pi Q^2}\left(1 - \frac{2b}{Q} + \frac{2b^2}{Q^2}\right) + \cdots,$$

$$S = \frac{A(b, Q)}{4}$$

$$= \pi Q^2\left(1 + \frac{2b}{Q} + \frac{2b^2}{Q^2}\right) + \cdots. \tag{21}$$

in (20) to get

$$E = \frac{3\pi Q^4}{nL} b\left(\frac{12\pi Q^3}{nL} - \frac{1}{4}\right) + b^2 . \frac{24\pi Q^2}{nL} + \cdots. \tag{22}$$

Comparison with the mass formula

$$M = Q + \frac{b^2}{2Q} + \ldots. \tag{23}$$

for the black hole shows that in general there is no agreement between E, M. However, for consistency, the *temperature* $\frac{dE}{dS}$, or equivalently $\frac{dE}{db}$ should vanish in the extremal limit when b vanishes. This leads to the condition

$$\frac{\pi Q^3}{nL} = \frac{1}{48}, \tag{24}$$

relating the product of the number n and L with the parameter Q characterizing the family of black holes being considered, and as a consequence the mass of the black hole and the energy of the gas agree for small values of b in the sense

$$E = M - \frac{15Q}{16} + O(b^3). \tag{25}$$

The difference between E and M represents a zero-point shift depending only on the charge of the black hole and independent of (small) b. The states of the gas of massless particles then can be regarded as a model for the *family* of near-extremal black holes with a fixed value of the charge but varying b as far as thermodynamics is concerned.

Although this is not a strictly string theoretic derivation of the area formula, it gives some of the flavour underlying the string calculation. More important, it indicates the limits of such calculations. The above agreement holds only for small b, because b^3 terms, which are nonvanishing, vitiate it. The charge Q cannot be taken to be zero because b has to be much smaller than it : Q^{-1} is in fact used in the calculation.

3.2 Further developments :

Various studies have been carried out to see how far these calculations can be linked to other black holes, especially the familiar Schwarzschild black hole. These involve the use of some symmetries of string theory. "U-duality" has been used to relate non-extremal 4 dimensional black holes to black holes with 3 noncompact dimensions for which Chern-Simons methods can be used to derive the entropy. The entropy is expected to be U-duality invariant and hence an indirect calculation becomes available for the non-extremal case. The results satisfy the standard area formula [10].

Combinations of "T-duality" and boosts followed by compactification have been used to relate non-extremal 4 dimensional black holes to extremal configurations for which direct counting methods are available. The expression $A/4G$ is kept unchanged in these transformations. The related systems are very different, but the boosts and compactifications are such that the horizons of the corresponding systems are related by coordinate transformations and may perhaps be imagined to be identical [11] !

4. Summary and conclusions

Black holes were originally regarded as interesting solutions of Einstein's classical theory of gravity. This theory does not lead to a straightforward quantum field theory, but semiclassical studies demonstrate the occurrence of *Hawking* radiation from black holes, which can be used to assign temperatures and (thermodynamic) entropies to these objects.

Hawking radiation, though a surprise in the beginning, was not really puzzling, in the way that the idea of a black hole entropy was. The question which has bothered everyone for a long time is about the identification of the quantum states responsible for the entropy.

Any quantum theory which contains semiclassical gravity in an approximation is expected to contain Hawking radiation and even a description of the states leading to the entropy.

String theory at low energy does contain Einstein's action, and indeed some progress has been made in identifying the states for some special supersymmetric black holes. But no general understanding has been achieved and it is premature to say that there is a full quantum theory of gravity. The agreement that has been found between string calculations and semiclassical expectations really means nothing more than that string theory contains semiclassical gravity as an approximation. What is interesting is that this theory, hitherto considered abstruse and inaccessible, has allowed these calculations to be done. It still lacks transparency, however.

The more direct approach of quantizing gravity has been successful in 2+1 dimensions but special techniques related to the specific dimensionality have played an important rôle. The extension of these techniques to 3+1 dimensions has made some progress, but there the result is spoilt by the appearance of an undetermined parameter.

It is to be hoped that the final resolution of the black hole entropy problem is not far away !

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